ON THE DENSITY OF THE DISTRIBUTION OF NATURAL FREQUENCIES OF THIN ELASTIC SHELLS

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The problem of finding the number of natural frequencies of a structure which lie within a given interval arises in connection with the study of the vibrations of structural elements under random loads having a wide spectrum [1]. Courant [2] considered this problem for membranes and plates. The problem is solved below for thin elastic shells whose vibrations can be described by equations valid for cases with large index of variation [3].

1. We shall consider a thin elastic shell of thickness h referred to orthogonal curvilinear coordinates x_1 and x_2 which coincide with the lines of principal curvature. Let R_1 and R_2 be the radii of curvature of the middle surface, E the modulus of elasticity, ρ the density of the material, D the plate stiffness, w the normal deflection, φ the stress function for the forces in the middle surface and ω the frequency of vibration. The equations for modes of vibration having sufficiently high indices of variation are of the form [3]

$$D\triangle\triangle w - \left(\frac{1}{R_2}\frac{\partial^2\varphi}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2\varphi}{\partial x_2^2}\right) - \rho h\omega^2 w = 0, \quad \frac{1}{Eh}\triangle\triangle\varphi + \left(\frac{1}{R_2}\frac{\partial^2 w}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2 w}{\partial x_2^2}\right) = 0 \tag{1.1}$$

Tangential inertia forces are not taken into account in equations (1.1). Therefore, only the frequencies of predominantly flexural vibrations can be found from these equations.

An asymptotic solution of equations (1.1) has been obtained [3, 4] for a rectangular region (in the generalized sense) with sides a_1 and a_2 within which the metric of the middle surface remains constant. The natural frequencies are determined from the formula

$$\omega^2 = \frac{D}{\rho h}\left[(k_1^2 + k_2^2)^2 + \frac{Eh}{DR_1^2}\,\frac{(k_1^2\chi + k_2^2)^2}{(k_1^2 + k_2^2)^2}\right] \qquad \left(\chi = \frac{R_1}{R_2}\right) \tag{1.2}$$

where the wave numbers k_1 and k_2 are found from the solution of the system of equations

$$\begin{aligned} k_1a_1 &= \tan^{-1} u_{11}(k_1, k_2) + \tan^{-1} u_{12}(k_1, k_2) + m_1\pi \\ k_2a_2 &= \tan^{-1} u_{21}(k_1, k_2) + \tan^{-1} u_{22}(k_1, k_2) + m_2\pi \quad (m_1, m_2 = 1, 2, \ldots) \end{aligned} \tag{1.3}$$

Here $u_{\alpha\beta}$ are functions of the wave numbers and depend on the boundary conditions. By the functions $\tan^{-1} u_{\alpha\beta}$ the principal values are meant. The solution (1.2) and (1.3) is always applicable except when the dynamic edge effect is degenerate. We have shown [3, 4] that for plates and spherical shells the dynamic edge effect never degenerates and the degeneracy occurs for cylindrical shells only for sufficiently small wave numbers

$$k_1^2 + k_2^2 \leqslant \left(\frac{Eh}{DR_1^2}\right)^{1/2}$$

For very large wave numbers ($k_1^2 + k_2^2 \sim h^{-2}$), the original equations (1.1) are no longer applicable since effects like shear deformation and rotatory inertia must then be taken into account. From formula (1.3), the cruder estimates

$$k_1a_1 = m_1\pi + O(1), \qquad k_2a_2 = m_2\pi + O(1) \qquad (m_1,\ m_2 = 1, 2, \ldots) \tag{1.4}$$

can be derived. These are analogous to the well-known estimates of Courant [2] for the natural frequencies of membranes and plates. It is essential to realize that the estimates (1.4) are valid only outside the range of degeneracy of the edge effect. This will be clear if it is recalled that from the physical point of view degeneracy implies a large effect of conditions at the edge on the mode shape within the region.

Let us construct the two families of curves (1.3) corresponding to various integral values of m_1 and m_2. The wave numbers are found as the coordinates of the points of intersection of these curves. If the shell is simply supported at its edge, all the $u_{\alpha\beta} = 0$. We then obtain a grid of straight lines parallel to the coordinate axes consisting of cells of dimensions $\Delta k_1 = \pi/a_1$ and $\Delta k_2 = \pi/a_2$ (in this case the asymptotic solution coincides with the exact one). It follows from the relation (1.3) that, in general, a change of the boundary conditions cannot displace the curves by more than the dimensions of one cell. This is also reflected in the formulas (1.4).

2. We shall now apply the relations (1.2) and (1.4) to obtain estimates of the density of distribution of the natural frequencies. Using an idea of Courant [2], we shall determine approximately the number of frequencies $N(\Omega)$ smaller than a given frequency Ω as the ratio of the area on the k_1, k_2 plane in which the inequality $\omega(k_1, k_2) < \Omega$ holds to the area of a single cell. It is apparent that this method of calculation becomes more reliable as the number of wave numbers lying in the region S (Fig. 1) bounded by the curve $\omega(k_1, k_2) = \Omega$ increases. We then have the formula

$$N(\Omega) \approx \frac{1}{\Delta k_1 \Delta k_2} \iint\limits_S dk_1 dk_2 \tag{2.1}$$

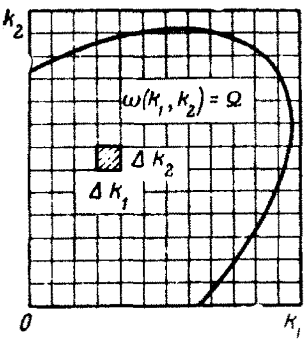

Fig. 1.

Using the notation

$$k_1^2 + k_2^2 = r^2, \quad \frac{k_2}{k_1} = \tan\theta, \quad \frac{1}{R_1}\left(\frac{E}{\rho}\right)^{1/2} = \Omega_R \tag{2.2}$$

Equation (1.2) can be written in the form

$$\omega^2 = \frac{D}{\rho h} r^4 + \Omega_R^2 (\chi \cos^2\theta + \sin^2\theta)^2 \tag{2.3}$$

From this result, after substitution into (2.1) and integration with respect to r, we obtain

$$N(\Omega) \approx \frac{a_1 a_2}{2\pi^2}\left(\frac{\rho h}{D}\right)^{1/2} \int\limits_{\theta_1(\Omega)}^{\theta_2(\Omega)} [\Omega^2 - \Omega_R^2 (\chi \cos^2\theta + \sin^2\theta)^2]^{1/2} d\theta \tag{2.4}$$

The integration with respect to θ is carried out over that part of the quadrant $0 \leqslant \theta \leqslant \pi/2$ in which the expression under the radical is positive. Differentiating the expression (2.4) with respect to Ω, we get an asymptotic formula for the density of the frequency distribution

$$\frac{dN(\Omega)}{d\Omega} \approx \frac{a_1 a_2}{2\pi^2}\left(\frac{\rho h}{D}\right)^{1/2} \Omega \int\limits_{\theta_1(\Omega)}^{\theta_2(\Omega)} \frac{d\theta}{[\Omega^2 - \Omega_R^2 (\chi \cos^2\theta + \sin^2\theta)^2]^{1/2}} \tag{2.5}$$

We now introduce the notation

$$\begin{aligned} H(\alpha, \chi) &= \frac{2}{\pi} \int\limits_{\theta_1(\alpha)}^{\theta_2(\alpha)} [1 - \alpha^2 (\chi \cos^2\theta + \sin^2\theta)^2]^{1/2} d\theta \\ H_1(\alpha, \chi) &= \frac{2}{\pi} \int\limits_{\theta_1(\alpha)}^{\theta_2(\alpha)} \frac{d\theta}{[1 - \alpha^2 (\chi \cos^2\theta + \sin^2\theta)^2]^{1/2}} \end{aligned} \qquad \left(\alpha = \frac{\Omega_R}{\Omega}\right) \tag{2.6}$$

Formulas (2.4) and (2.5) take the form

$$N(\Omega) \approx \frac{a_1 a_2}{4\pi}\left(\frac{\rho h}{D}\right)^{1/2} \Omega\, H\left(\frac{\Omega_R}{\Omega}, \chi\right), \qquad \frac{dN(\Omega)}{d\Omega} \approx \frac{a_1 a_2}{4\pi}\left(\frac{\rho h}{D}\right)^{1/2} H_1\left(\frac{\Omega_R}{\Omega}, \chi\right) \tag{2.7}$$

3. We introduce the further notation

$$\eta_{1,2} = \frac{1 \pm \alpha\chi}{\alpha(1-\chi)}, \quad \zeta_1 = \sqrt{\frac{2\alpha(1-\chi)}{(1+\alpha)(1-\alpha\chi)}}, \quad \zeta_2 = \sqrt{\frac{2\alpha(\chi-1)}{(1-\alpha)(1+\alpha\chi)}}$$

By means of the substitution $\sin^2\theta = \xi$, the second of the integrals (2.6) can be reduced to the form

$$H_1(\alpha, \chi) = \frac{1}{\alpha\pi|1-\chi|}\int_{\xi_1}^{\xi_2} \frac{d\xi}{[\xi(1-\xi)(\eta_1+\xi)(\eta_2-\xi)]^{1/2}} \tag{3.1}$$

The integration is carried out over the part of the interval $0 \leqslant \xi \leqslant 1$ in which the expression under the radical is positive.

For $\chi < 1$ the following four cases are possible.

a) $\eta_1 > 0$, $\eta_2 < 1$. In this case $\xi_1 = 0$, $\xi_2 = 1$ and formula (3.1) takes the form

$$H_1(\alpha, \chi) = \frac{2}{\pi\sqrt{(1+\alpha)(1-\alpha\chi)}} K(\zeta_1) \tag{3.2}$$

where $K(\zeta_1)$ is the complete elliptic integral of the first kind.

b) $\eta_1 > 0$, $\eta_2 < 1$. In this case $\xi_1 = 0$, $\xi_2 = \eta_2$ and, therefore

$$H_1(\alpha, \chi) = \frac{\sqrt{2}}{\pi\sqrt{\alpha(1-\chi)}} K(\zeta_1^{-1}) \tag{3.3}$$

c) $\eta_1 < 0$, $\eta_2 > 1$. Here $\xi_1 = -\eta_1$, $\xi_2 = 1$; after some transformations we arrive at formula (3.3).

d) $\eta_1 < 0$, $\eta_2 < 1$. In this case $\xi_1 = -\eta$, $\xi_2 = \eta_2$ and we obtain formula (3.2).

Now let $\chi > 1$. Two cases are possible here:

a) $\alpha\chi < 1$. Then $\xi_1 = 0$, $\xi_2 = 1$ and formula (3.1) gives

$$H_1(\alpha, \chi) = \frac{2}{\pi\sqrt{(1-\alpha)(1+\alpha\chi)}} K(\zeta_2) \tag{3.4}$$

b) $\alpha\chi > 1$. Then $\xi_1 = \eta_1$, $\xi_2 = 1$, and, therefore

$$H_1(\alpha, \chi) = \frac{\sqrt{2}}{\pi\sqrt{\alpha(\chi-1)}} K(\zeta_2^{-1}) \tag{3.5}$$

4. Two special cases deserve attention. For a spherical shell $\chi = 1$ and the integral (2.6) can be expressed at once in terms of elementary functions

$$H_1(\alpha, 1) = 0 \quad (\alpha > 1) \tag{4.1}$$

$$H_1(\alpha, 1) = \frac{1}{\sqrt{1-\alpha^2}} \quad (\alpha < 1),$$

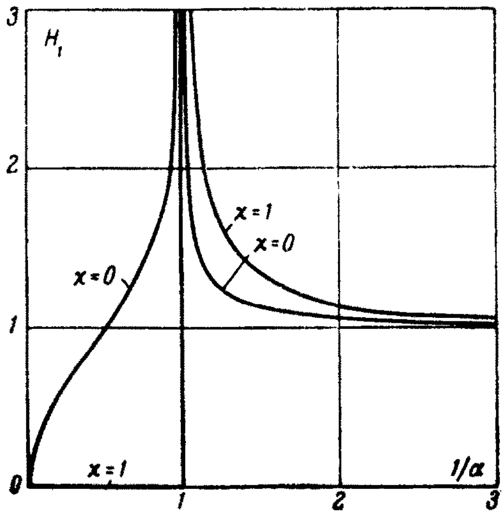

Fig. 2.

Thus, the density of the frequencies is equal to zero for $\Omega < \Omega_R$. It has a singularity for $\Omega = \Omega_R$ (in the present case Ω_R is the minimum frequency of vibration). The density of the frequency distribution rapidly approaches the density for the corresponding plate for $\Omega \gg \Omega_R$.

In the case of a cylindrical shell ($\chi = 0$) we find, in accordance with formulas (3.2) and (3.3)

$$\begin{aligned} H_1(\alpha, 0) &= \frac{2}{\pi\sqrt{1+\alpha}} K\left(\sqrt{\frac{2\alpha}{1+\alpha}}\right) \qquad (\alpha < 1) \\ H_1(\alpha, 0) &= \frac{\sqrt{2}}{\pi\sqrt{\alpha}} K\left(\sqrt{\frac{1+\alpha}{2\alpha}}\right) \qquad (\alpha > 1) \end{aligned} \tag{4.2}$$

The results of calculations based on formulas (4.1) and (4.2) are shown in Fig. 2.

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